

Second Generation of Moore–Read Quasiholes in a Composite Fermion Liquid

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Two- and three-body correlations of incompressible quantum liquids are studied numerically. Pairing of composite fermions (CFs) in the $1/3$ -filled second CF Landau level is found at $\nu_e = 4/11$. It is explained by reduced short-range repulsion due to ring-like single-particle charge distribution. Although Moore–Read state of CFs is unstable in the $1/2$ -filled second CF level, condensation of its quasiholes is a possible origin of incompressibility at $\nu_e = 4/11$. Electron pairing occurs at $\nu_e = 7/3$ and $13/3$, but with different pair–pair correlations. Signatures of triplets are found at higher fillings.

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Strong magnetic field B applied to a two-dimensional electron gas (2DEG) rearranges its single-particle density of states to a series of discrete Landau levels (LL_{*n*}). When the cyclotron gap $\hbar\omega_c \propto B$ exceeds Coulomb energy $e^2/\lambda \propto \sqrt{B}$ ($\lambda = \sqrt{\hbar c/eB}$ being the magnetic length), the low-energy dynamics depends on interactions in one, partially filled LL. Despite reminiscence to atomic physics, macroscopic degeneracy and a distinct scattering matrix lead to very different, fascinating behavior [1].

Fractional quantum Hall effect [2] reveals plethora of highly correlated electron phases at various LL filling factors $\nu_e = 2\pi\varrho\lambda^2$ (ϱ being the 2D concentration). Among them are Laughlin [3] and Jain [4] incompressible liquids (IQLs) with fractionally charged quasiparticles (QPs) at $\nu_e = \frac{1}{3}$ or $\frac{2}{5}$, Wigner crystals [5] at $\nu_e \ll 1$, and stripes [6] in high LLs. Besides transport [2], they are probed by shot-noise (allowing detection of fractional charge of the QPs [7]) and optics (with discontinuities in photoluminescence energy related to the QP interactions [8]).

A key concept in understanding IQLs is Jain’s composite fermion (CF) picture [4]. The CFs are fictitious particles, electrons that captured part of the external magnetic field B in form of infinitesimal tubes carrying an even number $2p$ of flux quanta $\phi_0 = hc/e$. The most prominent IQLs at $\nu_e = n(2ps \pm 1)^{-1}$ are represented by the completely filled s lowest LLs of the CFs (CF-LL_{*n*} with $n < s$) in a residual magnetic field $B^* = B - 2p\phi_0\varrho$.

Not all IQLs are so easily explained by the CF model, e.g., Haldane–Rezayi [9] and Moore–Read [10] paired liquids proposed for $\nu_e = \frac{5}{2}$. Because of nonabelian statistics of its quasiholes (QHs), especially the latter state has recently stirred renewed interest as a candidate for quantum computation in a solid-state environment [11].

Another family of IQLs discovered by Pan *et al.* [12] at $\nu_e = \frac{4}{11}$, $\frac{3}{8}$, and $\frac{5}{13}$ corresponds to fractional CF fillings $\nu_{\text{CF}} = \nu_e(1 - 2p\nu_e)^{-1} = \frac{4}{3}$, $\frac{3}{2}$, and $\frac{5}{3}$ (with $p = 1$). Assuming spin polarization, all these states have a partially filled CF-LL₁. Their incompressibility results from residual CF–CF interactions. Familiar values of ν_{CF} suggested similarity between partially filled electron and CF LLs [13]. For $\nu_e = \frac{4}{11}$ and $\frac{5}{13}$, it revived the “QP hierarchy” [14], whose CF formulation consists of the CF \rightarrow electron

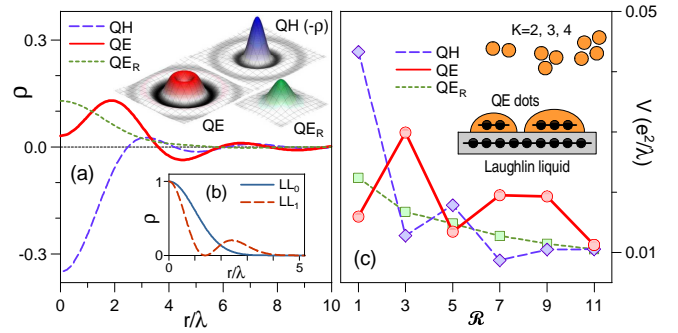


FIG. 1: (color online). (a) Radial charge distribution profiles of different composite fermions: Laughlin quasielectron (QE), quasihole (QH), and reversed-spin quasielectron (QE_R); results obtained from exact 10-electron diagonalization; λ is the magnetic length. (b) Same for electrons in two lowest Landau levels. (c) Haldane pseudopotentials (interaction energy V vs. relative pair angular momentum R) for composite fermions; inset: schematic of “artificial composite fermion atoms.”

mapping followed by reapplication of the CF picture in CF-LL₁, leading to a “second generation” of CFs [15]. However, this idea ignored the requirement of a strong short-range repulsion [16, 17]. Indeed, it was later excluded in exact diagonalization studies [18], in which a different series of finite-size $\nu_e = \frac{4}{11}$ liquids with larger gaps was identified. On the other hand, Moore–Read liquid of paired CFs was tested [19] for $\nu_e = \frac{3}{8}$, but it was eventually ruled out in favor of the stripe order [20, 21].

In this Letter, we study two- and three-body correlations in several IQLs whose origin of incompressibility remains controversial. We find evidence for CF pairing in the $\nu_e = \frac{4}{11}$ liquid, hence interpreted as a condensate of (nonabelian) QHs of the “second generation” Moore–Read state of the CFs. The pair–pair or QH–QH correlations are not defined, but a Laughlin form is excluded.

In Fig. 1(a,b) charge-density distributions of electrons are compared with three different CF quasiparticles at $\nu_e = \frac{1}{3}$. Laughlin liquid is a filled spin-polarized CF-LL₀, and its quasielectron (QE), quasihole (QH), and reversed-spin quasielectron (QE_R) correspond to a particle in CF-LL₁, a vacancy in CF-LL₀, and a spin-flip particle in CF-

LL₀, respectively. Particles/holes in CF-LL₀ resemble those in LL₀. However, the ring structure in CF-LL₁ makes the QEs different from the electrons and causes strong reduction of the QE-QE repulsion at short range [cf. Fig. 1(c)]. Such interaction cannot [16, 17] produce a Laughlin IQL of the QEs at the $\nu = \frac{1}{3}$ filling of CF-LL₁. Instead, other QE-QE correlations must be considered.

Spontaneous QE cluster formation would be somewhat analogous to the self-assembled growth of strained quantum dots [22]. A full CF-LL₀ representing the uniform-density Laughlin liquid plays the role of a “wetting layer.” Over this background, in analogy to atoms grouping into dots to minimize the elastic energy, QEs moving within CF-LL₁ arrange themselves into pairs or larger QE clusters easily pinned down by disorder. While in electronic “artificial atoms” the self-organization of real atoms serves a purpose of external confinement for the electrons, in their CF analogs both these roles are played by the QEs. Another distinction is the fractional charge of bound QE carriers. A similar *electron*-atom analogy was earlier explored for condensation of cold atoms [23].

In numerics we considered $N \leq 12$ particles ($N = 12$ being divisible by $K = 2, 3$, and 4) of charge q ($-e$ for electrons and $-\frac{1}{3}e$ for the CFs) confined to a Haldane sphere [14] of radius R . For its high symmetry, this geometry is especially useful in studying quantum liquids. The radial magnetic field B is created by a Dirac monopole of strength $2Q = 4\pi R^2 B \phi_0^{-1}$. The single-particle LLs are distinguished by shell angular momentum $l \geq Q$.

As for a partially filled atomic shell, the many-body hamiltonian on a sphere is determined by particle number N , shell degeneracy $g = 2l + 1$, and interaction matrix elements. Using Clebsch-Gordan coefficients, the latter are related to Haldane [24] pseudopotentials V_L (energies of pairs with angular momentum L). The pseudopotential combines information about the potential $v(r)$ and shell wavefunctions, so it may not be similar in different systems with the same (Coulomb) forces. In macroscopic quantum Hall systems, only the ratio $\nu = N/g$ (filling factor) is important, and V is a function of relative pair angular momentum $\mathcal{R} = 2l - L$ (for fermions, an odd integer). The strategy in exact diagonalization is therefore to study different finite systems ($N, 2l$) with a realistic interaction $V(\mathcal{R})$, in search of those properties which scale properly with size and persist in the macroscopic limit.

In the following we will distinguish $\nu_e = 2\pi\phi\lambda^2$ from the effective filling factor $\nu = N/g < 1$ of only those electrons or CFs in their highest, partially filled shell. In LL _{n} , $\nu_e = 2n + \nu$. In CF-LL _{n} (assuming spin-polarization) $\nu_{\text{CF}} = n + \nu$ and $\nu_e = \nu_{\text{CF}}(2p\nu_{\text{CF}} + 1)^{-1}$.

The CF pseudopotentials shown in Fig. 1(c) were obtained using a similar method to Ref. [20], by combining short-range data from exact diagonalization [17] with long-range behavior of point charges $\pm\frac{1}{3}e$. Weak QE-QE repulsion at $\mathcal{R} = 1$ is the reason why the $\nu = \frac{1}{3}$, $\frac{2}{3}$, and $\frac{1}{2}$ states of QEs are not the “second generation”

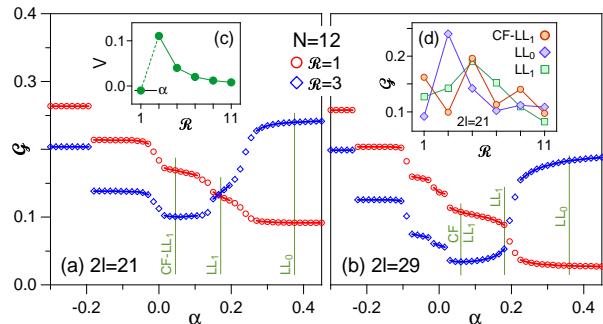


FIG. 2: (color online). Haldane pair amplitudes \mathcal{G} (\sim number of pairs) at relative pair angular momenta $\mathcal{R} = 1$ and 3, of $N = 12$ fermions in angular momentum shells with $2l = 21$ (a) and $2l = 29$ (b), as a function of parameter α of the interaction pseudopotential shown in (c). (d) Amplitudes $\mathcal{G}(\mathcal{R})$ of electrons and composite fermions in different Landau levels.

Laughlin, Jain, or Moore-Read states (of QEs). The average QE-QE interaction energies (per particle) in these states overestimates the actual QE eigenenergies by at least $0.003 e^2/\lambda$ (6–7%). Clearly, the microscopic origin of the observed QE incompressibility must be different.

What are these known correlations, excluded for QEs? Laughlin correlations result from strong short-range repulsion (such as between electrons in LL₀). They consist of the maximum avoidance of pair states with the smallest \mathcal{R} . E.g., Laughlin $\nu = \frac{1}{3}$ state is the zero-energy ground state of a model pseudopotential $V = \delta_{\mathcal{R},1}$ [14]. For more realistic interactions, the exact criterion is that V must rise faster than linearly when \mathcal{R} decreases [17]. A linear decrease of V between $\mathcal{R} = 1$ and 5 (such as in LL₁) leads to different correlations. E.g., Moore-Read $\nu = \frac{1}{2}$ liquid involves pairing and Laughlin correlations among pairs. It is the zero-energy ground state of a model three-body pseudopotential $V = \delta_{\mathcal{T},3}$ ($\mathcal{T} = 3l - L \geq 3$ is the relative triplet angular momentum, proportional to the area spanned by three particles) [10].

Weak QE-QE repulsion at $\mathcal{R} = 1$ compared to $\mathcal{R} = 3$ could force QEs into even larger clusters. As a simple classical analogy, consider a string of point particles, one per unit length, with a repulsive potential $v_a(r) = a + (1 - a)r$ for $r < 1$ and $1/r^2$ otherwise. Equal spacing is favored for $a > 1.64$, and transitions to pairs, triplets, and larger clusters occur for decreasing a . A similar rearrangement might occur when going from LL₀ to LL₁ and CF-LL₁, with $V(1)$ playing the role of $v_a(0) \equiv a$.

In Fig. 2 we plot two leading “Haldane amplitudes” [24], $\mathcal{G}(1)$ and $\mathcal{G}(3)$. The discrete pair-correlation function $\mathcal{G}(\mathcal{R})$ is proportional to the number of pairs with a given \mathcal{R} and normalized to $\sum_{\mathcal{R}} \mathcal{G}(\mathcal{R}) = 1$. It connects many-body interaction energy with a pseudopotential, $E = \binom{N}{2} \sum_{\mathcal{R}} \mathcal{G}(\mathcal{R}) V(\mathcal{R})$. Here, \mathcal{G} is calculated in the ground states of $N = 12$ particles at $2l = 21$ and 29 (corresponding to $\nu = \frac{1}{2}$ and $\frac{1}{3}$ for the QEs [18]) with

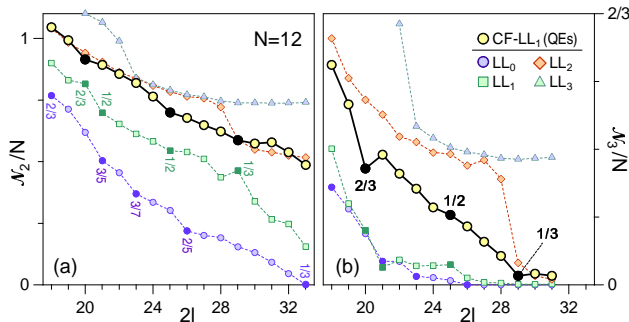


FIG. 3: (color online). Number of pairs \mathcal{N}_2 (a) and triplets \mathcal{N}_3 (b) with the minimum relative angular momentum ($\mathcal{R} = 1$ or $\mathcal{T} = 3$) for $N = 12$ electrons or composite fermions in angular momentum shells with $2l = 18$ to 33 , corresponding to fractional Landau level fillings $\frac{1}{3} \leq \nu \sim N/(2l+1) \leq \frac{2}{3}$. Finite-size incompressible states are labeled by ν .

model interaction shown in the inset: $V_\alpha(1) = \alpha$ and $V_\alpha(\mathcal{R} > 1) = 1/\mathcal{R}^2$. At $\alpha > 0.3$, $\mathcal{G}(1)$ takes on the minimum possible value, which means Laughlin correlations (no clusters). At $\alpha < -0.25$, $\mathcal{G}(1)$ reaches maximum, and the particles form one big $\nu = 1$ quantum Hall droplet (QHD). The transition between the two limits occurs quasi-discontinuously through a series of well-defined states seen as plateaus in $\mathcal{G}(\alpha)$.

The cluster size K cannot be assigned to each state because the number of plateaus depends on the choice of V_α . The comparison of $\mathcal{G}(1)$ with the values predicted for N/K independent QHDs of size $K = 2, 3$, and 4 is not convincing because in a few-cluster system each QHD is relaxed by the cluster-cluster interaction, lowering $\mathcal{G}(1)$. Another problem is the contribution to $\mathcal{G}(1)$ from pairs of particles belonging to different clusters. Nevertheless, it is clear that the “degree of clustering” changes as a function of α in a quantized fashion, supporting the picture of N particles grouping into various clustered configurations. Furthermore, the values of α for which V_α reproduces the exact ground states of QEs or electrons belong to different continuity regions, confirming different correlations in LL0, LL1, and CF-LL1 (except for a possible similarity of the $\nu = \frac{1}{3}$ states in LL1 and CF-LL1).

In Fig. 3(a) we compare $\mathcal{N}_2 = \binom{N}{2} \mathcal{G}(1)$, the number of pairs with $\mathcal{R} = 1$, calculated in the ground states of $N = 12$ CFs and electrons as a function $2l$. The downward cusps in $\mathcal{N}_2(2l)$ at a series of Laughlin/Jain states in LL0 are well understood. We also marked $2l = 2N - 3$ and $3N - 7$ corresponding to incompressible $\nu = \frac{1}{2}$ and $\frac{1}{3}$ ground states in LL1 and CF-LL1 [18], and their particle-hole conjugates ($N \rightarrow g - N$) at $2l = 2N + 1$ and $\frac{3}{2}N + 2$.

The comparison of \mathcal{N}_2 tells about short-range pair correlations in different LLs. There are significantly more pairs in CF-LL1 and in excited electron LLs than in LL0. In LL1, the Moore-Read state is known to be paired, and indeed $\mathcal{N}_2 \approx \frac{1}{2}N$ at $\nu = \frac{1}{2}$. A similar value is obtained

for the (not well understood) $\nu = \frac{1}{3}$ state at $2l = 29$. The CF-LL1 is different (in terms of \mathcal{N}_2) from LL0 or LL1 in the whole range of $18 \leq 2l \leq 33$. However, it appears similar to LL2 at both $2l \leq 23$ and $2l \geq 29$. Also, LL2 and LL3 look alike for $23 \leq 2l < 29$. While convincing assignment of ν to a finite state ($N, 2l$) requires studying size dependence (we looked at different $N \leq 12$), notice that $N/g = \frac{1}{2}$ at $2l = 23$, and $2l = 29$ is the $\nu = \frac{1}{3}$ state in LL1 and CF-LL1. Note also that similar short-range correlations do not guarantee high wavefunction overlaps. Here, only $\langle \text{LL}_2 | \text{LL}_3 \rangle^2$ reaches 0.67 while all other overlaps, including $\langle \text{QE} | \text{LL}_n \rangle^2$, essentially vanish.

In Fig. 3(b) we plot \mathcal{N}_3 , the number of “compact” triplets with $\mathcal{T} = 3$. It is proportional to the first triplet Haldane amplitude and tells about short-range three-body correlations. In both LL0 and LL1, \mathcal{N}_3 decreases roughly linearly as a function of $2l$ and drops to essentially zero at $2l = 21$, the smallest value at which the $\mathcal{T} = 3$ triplets can be completely avoided. Exactly $\mathcal{N}_3 = 0$ would indicate the Moore-Read state, but its accuracy for the actual $\nu = \frac{1}{2}$ ground state in LL1 depends sensitively on the quasi-2D layer width and on the surface curvature. Nevertheless, clusters larger than pairs clearly do not form in neither LL0 nor LL1 at $\nu \leq \frac{1}{2}$.

The number of QE triplets in CF-LL1 is also a nearly linear function of $2l$, but it drops to zero at $2l = 3N - 7 = 29$, earlier identified with $\nu = \frac{1}{3}$ in this shell (i.e. with $\nu_e = \frac{4}{11}$) [18]. In connection with having $\mathcal{N}_2 \approx \frac{1}{2}N$ pairs, *vanishing of \mathcal{N}_3 is the evidence for QE pairing at $\nu_e = \frac{4}{11}$.*

The elementary excitations that appear in the paired $\nu = \frac{1}{2}$ Moore-Read state when $2l > 2N - 3$ are the $\frac{1}{4}q$ -charged QHs (of the Laughlin liquid of pairs) and pair-breaking neutral-fermion excitations [10]. Being paired, the QE state at $2l = 3N - 7$ *can only contain the QHs but no pair-breakers*. The interaction of Moore-Read QHs in CF-LL1 is not known, but evidently it causes their condensation into an incompressible liquid at $\nu = \frac{1}{3}$.

The “second generation” (to distinguish from $\nu_e = \frac{5}{2}$) Moore-Read state of QEs would occur at $\nu = \frac{1}{2}$ in CF-LL1 (i.e., at $\nu_{\text{CF}} = \frac{3}{2}$ or $\nu_e = \frac{3}{8}$). Its instability [20, 21] does not preclude reentrance with additional QHs at a lower ν and, in particular, their condensation at $\nu = \frac{1}{3}$ (i.e., at $\nu_{\text{CF}} = \frac{4}{3}$ or $\nu_e = \frac{4}{11}$). A similar situation occurs with Jain $\nu = \frac{2}{7}$ state, obtained (in Haldane hierarchy) from Laughlin $\nu = \frac{1}{3}$ state by addition of “second generation” Laughlin QHs. There, stability of the $\nu = \frac{1}{3}$ daughter does *not* require stability of the $\nu = \frac{1}{3}$ parent.

The value of $2l = 3N - 7$ precludes a Laughlin state of pairs (or, equivalently, of the QHs). To show it, let us use the following pictorial argument, equivalent to a more rigorous derivation. Laughlin $\nu = \frac{1}{3}$ state (of individual particles) can be viewed as $\bullet \circ \circ \bullet \dots \bullet \circ \circ \bullet \equiv (\bullet \circ \circ) \bullet$, with “ \bullet ” and “ \circ ” denoting particles and vacancies. Counting the total LL degeneracy g leads to the correct value of $2l = 3N - 3$. The Moore-Read state, i.e., the Laughlin state of pairs at $\nu = \frac{1}{2}$, is represented by

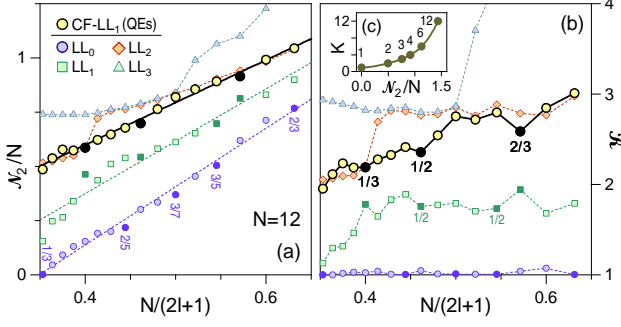


FIG. 4: (color online). Number of pairs \mathcal{N}_2 with the minimum relative angular momentum $\mathcal{R} = 1$ (a) and estimated average cluster size \mathcal{K} (b) for $N = 12$ electrons or composite fermions in Landau levels angular momentum shells with $2l = 18$ to 33 , plotted as a function of the filling factor $\nu \sim N/(2l+1)$.

$(\bullet\bullet\circ\circ)\bullet\bullet$, yielding $2l = 2N - 3$. A Laughlin state of pairs at $\nu = \frac{1}{3}$ would correspond to $(\bullet\bullet\circ\circ\circ)\bullet\bullet$, predicting (incorrectly) $2l = 3N - 5$. Assuming pairing, $2l = 3N - 7$ can only be obtained using a two-pair unit cell, $(\bullet\bullet\circ\circ\bullet\bullet\circ\circ\circ\circ)\bullet\bullet\circ\circ\bullet\bullet$, corresponding to more complicated pair-pair correlations.

At higher fillings of CF-LL₁, $\mathcal{N}_3 \approx \frac{1}{3}N$ at $2l = 20$ suggests division of N QEs into $\frac{1}{3}N$ triplets at $\nu = \frac{2}{3}$, and $\mathcal{N}_3 \approx \frac{1}{6}N$ at $2l = 25$ implies a more complicated cluster configuration (with mixed sizes) at $\nu = \frac{1}{2}$. LL₂ and LL₃ look alike (and different from LL₀ or CF-LL₁) at $23 \leq 2l < 29$, both having $\mathcal{N}_3 \approx \frac{1}{3}N$. At $2l = 29$, \mathcal{N}_3 for LL₂ drops rapidly to almost zero. This further supports similarity of the $\nu = \frac{1}{3}$ states in LL₂ and CF-LL₁.

In Fig. 4(a) we replot \mathcal{N}_2 as a function of $N/g \sim \nu$. The quasi-linear dependences for LL₀, LL₁, and CF-LL₁ all aim correctly at $\mathcal{N}_2 = 2N - 3$ for $\nu = 1$, but start from different values, $\mathcal{N}_2 \approx 0$, $\frac{1}{4}N$, and $\frac{1}{2}N$, at $\nu = \frac{1}{3}$. Regular dependence allows subtraction from \mathcal{N}_2 the contribution from pairs belonging to different clusters. As reference we used ground states of $V = \delta_{\mathcal{R},1}$. This short-range repulsion guarantees maximum avoidance of $\mathcal{R} = 1$; its \mathcal{N}_2^* contains only the inter-cluster contribution. To compare \mathcal{N}_2 of QEs or electrons with \mathcal{N}_2^* , we: (i) calculated \mathcal{N}_2 for a single K -size cluster, and multiplied it by N/K to obtain relation between \mathcal{N}_2 and K in an idealized clustered state of N particles, (ii) using this relation [cf. Fig. 4(c)], converted \mathcal{N}_2 and \mathcal{N}_2^* into the (average) cluster sizes K and K^* ; (iii) defined $\mathcal{K} = K - (K^* - 1)$ as the cluster size estimate free of the inter-cluster contribution.

The result in Fig. 4(b) indicates pairing in LL₁ at $\frac{1}{3} \leq \nu \leq \frac{2}{3}$, and in both CF-LL₁ and LL₂ at $\nu \leq \frac{1}{3}$. Triplets seem to form in CF-LL₁ at $\nu = \frac{2}{3}$, in LL₂ at $\frac{1}{3} \leq \nu \leq \frac{2}{3}$, and in LL₃ at $\nu \leq \frac{1}{2}$. The $\nu = \frac{1}{2}$ state of QEs falls between $\mathcal{K} = 2$ and 3 , suggesting mixed-size clusters.

In conclusion, we studied two- and three-body correlations of several quantum liquids. We found evidence for pairing of CFs at $\nu_e = \frac{4}{11}$ and interpret this state as a condensate of “second generation” Moore–Read QHs.

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